DYNAMIC RESPONSE OF RAYLEIGH BEAM ON WINKLER FOUNDATION SUBJECTED TO PARTIALLY DISTRIBUTED MOVING LOAD

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ABSTRACT

Dynamic response of Euler-Bernoulli beam subjected to concentrated moving load was investigated in this paper. The governing equation of fourth order partial differential equation was reduced to an ordinary differential equation using Series solution. The reduced second order differential equation was then solved using finite difference method. Numerical result was presented and it is found that the dynamic response of the beam initially moves in a steady state before deflecting and the deflected amplitude increases as the axial force, distance covered by the load, Mass of the load, speed at which the load moves increases but decreases as the length of the load, coefficient of the foundation increases. This study also conclude that R_0 has no/little effects on the structure of the beam.

Keywords: Amplitude, Axial Force, Concentrated Load, Euler-Bernoulli Beam, Finite Difference Method,

INTRODUCTION

In the recent years all branches of transport have experienced great advances characterized by increasing higher speeds and weight of vehicles (Awodola 2005). As a result, structures and media over or in which the vehicles move have been subjected to vibrations and dynamic stresses far larger than ever before. Many scholars have studied vibration of elastic and inelastic structures under the action of moving loads for many years, and effort are still being made to carry out investigation dealing with various aspect of the problem (Awodola 2007). The structures on which these moving loads are usually modeled are by elastic beams, plates or shells. The problem of elastic beam under the action of the moving loads was considered. Willis (1951) made the assumption that the mass of the beams is smaller than that of the load and obtained an approximate solution of the problem. Yoshida (1971) studied the vibration of a beam subjected to moving concentrated moving using finite element method. A simple beam supported to a constant moving force at uniform speed was considered. Krylov (1995) used the method of expansion of the associated Eigen modes. He assumed the mass of the load to be smaller than that of the beam. Bolotin (1964) carried out a dynamic analysis of the problem involving a concentrated mass traversing a simply supported beam at constant speed. His approach involves using Galerkin's method. The response of finite simply supported Euler-Bernoulli beam to a unit force moving at a uniform velocity was investigated by (Lee 1994). The effects of this moving force on beams with and without an elastic foundation were analyzed. In all the studies discussed above it was only the force effect of the moving loads that was taken into consideration. The moving load problem involving both the inertia effect as well as the force effects were not considered for several years. This type of dynamical problem was first considered by (Kalker 1993), later by (Jeffcott 2000) whose iterative method became divergent in some cases. Recently, (Esmailzadeh et.al 1995) worked on the vibration analysis of beams traversed by uniform partially distributed moving masses using analytical-numerical method. They discovered that the inertia effect of the distributed moving mass is of importance in the dynamic behaviour of the structure. The critical speeds of the moving load were also calculated for the mid span of the beam.

The length of the distributed moving mass was also found to affect the dynamic response. The effects of the speed of the moving load, the foundation stiffness and the length of the beam on the response of the beam have been studied and dynamic amplifications of deflection and stress have been evaluated. Based on the Langragian approach, (Chang 2000) analyzed the vibration of a multi-span non-uniform bridge subjected to a moving vehicle by using modified beam vibration functions as the assumed modes. The vehicle is modeled as a two-degree of freedom system. Obtained results are presented in form of dynamic amplification factors and compelled published results where applicable. The investigation into the dynamic response of a Bernoulli beam resting on Winkler foundation under the action of uniform partially distributed moving load was presented by (Usman 2003). Finite difference method was used to solve the coupled partial differential equation, result revealed that the amplitude of the beam resting on Winkler foundation increases with an increase in the value of the foundation constant. (Usman 2013) investigated the dynamic behavior of Euler-Bernoulli beam with structural damping subjected to partially distributed moving loads. Analytical numerical method was used to solve the governing equation, it was observed from the result that the damping increased with increase in resultant solution at constant fixed length of the beam. (Savin 2001) derived analytic expressions of the dynamic amplification factor and the characteristics response spectrum for weakly damped beams with various boundary conditions subjected to point loads moving at constant speeds. The obtained coefficients are given as functions of the ratio of the span length to the loads wavelength and the loads wavelength respectively. (Pesterve et.al,2000) developed simple tools for finding the maximum deflection of a beam for any given velocity of the traveling force. It is shown that for given boundary conditions, there exists a unique response-velocity dependence function. They suggested a technique to determine this function which is based on the assumption that the maximum beam response can be adequately approximated by means of the first mode. Also the maximum response function is calculated analytically for a simply supported beam and constructed numerically for a clamped-damped beam. Friction dampers are another common passive vibration control systems which dissipate energy through friction forces. These forces are generated with moving parts by sliding over each other. The energy dissipated by a friction damper reduces the energy demand on the structure and damps the structural response. The friction damper system includes the friction unit and a structural system in order to integrate the friction unit with the structure. The structural system can be either steel braces bolted to corner regions of the open bay space in the frame or an infill wall with gaps around the edges to prevent stiffness interaction of the wall with the frame members. Friction dampers are used as sacrificial or non-sacrificial elements. Their utilization as sacrificial elements is a very common attitude in civil engineering environment. In earthquake engineering applications, some of the structural members might be sacrificed in order to prevent the collapse of entire structure. These structural members absorb and dissipate the transmitted energy through plastic deformation in specially detailed regions. Location of the friction damper and stiffness of the braces which are used in order to install dampers are the main factors that affect the design parameters of the damper. (Nguyen 2011) (Dahlberg 1999) uses the modal analysis technique to investigate the influence of modal cross-spectral densities on the spectral densities of some responses of simply supported beams. The random response of damped beams was studied by (Jacquot 2000). The author presents a method of vibration analysis using the response power spectral density function and mean square response of considered beam structures excited by a second stationary random process. (Kukla et.al, 1993) dealt with the random vibration of a clamped-pinned beam. The flux of energy which is emitted by the vibrating beam was investigated. (Papadimitriou et.al 2005) provide a methodology for optimal establishment of the number and location of sensors on randomly vibrating structures for the purpose of the response predictions at unmeasured locations in structural systems. The author referees the results of considerations to randomly vibrating beams and plates. Its well known that damping becomes important when the need to have a thorough understanding of the control and mechanical response of vibrating structures arises. The problem of determining the dynamic response of a rectangular, damped, elastic plate carrying uniform partially moving load is investigated. The elastic plate is assumed to have uniform cross-sectional area. The effect of both rotatory inertia as well as shear deformation is assumed negligible. The moving partially distributed load is also assumed moving at uniform velocity. A constant damping coefficient is used throughout the analysis. Viscous damping whose coefficient is assumed to be directly proportional to the mass distribution of the system is considered in (Gbadeyan et.al 2002). An asymptotic analysis of eigen frequencies of uniform beam with both structural and viscous damping coefficient has also been carried out in(Hankum et.al 1991) and (Huang 1985).

Furthermore, (Kenny 1954) took up the problem of investigating the dynamic response of infinite elastic beams on elastic foundation when the beam is under the influence of a dynamic load moving with constant speed. Lie included the effects of viscous damping in the governing differential equation of motion. More recently, (Oni 1991) considered the problem of a harmonic time variable concentrated force moving at a uniform velocity over a Unite deep beam. The methods of integral transformations are used. In particular, the Unite Fourier transform is used for the length coordinate and the Laplace transform the time coordinate. Series solution, which converges as obtained for the deflection of simply supported beams. The analysis of the solution was carried out for various speeds of the load.

(Awodola 2007) worked on the influence of foundation and axial force on the vibration of a simply supported thin (Bernoulli Euler) beam, resting on a uniform foundation, under the action of a variable magnitude harmonic load moving with variable velocity is investigated in the paper. The governing equation is a fourth order partial differential equation. For the solution of this problem, in the first instance, the finite Fourier sine transformation is used to reduce the equation to a second order partial differential equation. The reduced equation is then solved using the Laplace transformation. Numerical analysis shows that the transverse deflection of the thin beam, resting on a uniform foundation, under the action of a variable magnitude harmonic load moving with variable velocity decreases as the foundation constant increases. It also shows that as the axial force increases, the transverse deflection of the thin beam decreases. Furthermore, (Milormir, et.al, 1969) developed a theory describing the response of a Bernoulli-Euler beam under an arbitrary number of concentrated moving masses. The theory is based on the Fourier technique and shows that, for a simply supported beam, the resonance frequency is lower with no corresponding decrease in maximum amplitude when the inertia is considered. (Usman et al 2018) presented an analysis of free vibrations of a cantilever beam and simply supported beam using series solution. It was found that the Deflection of beam increases as the length of the beam increases for a cantilever beam but decreases for the case of a simply supported beam. The response amplitude of a cantilever beam is greater than that of a simply supported beam.

This study seeks to analyse the dynamic effect of vibration of Euler-Bernoulli beam on Winkler foundation subjected to concentrated load using series solution method. In order to achieve the set aim, the following are the objectives of this project work, which are:

1. To theoretically represent Euler-Bernoulli beam subjected to concentrated load in the form of a fourth order Partial Differential equation.

2. To find the analytical solution of the governing partial differential equation of the beam
3. To determine the deflection of the beam subject to the initial and boundary condition of the system.

4. To graphically represent the dynamic response of the deflection of the beam.

MATHEMATICAL FORMULATION

Consider a non-prismatic Rayleigh beam with damping coefficient of length L resting on a Winkler foundation and transverse by uniform partially distributed moving mass. The resulting vibrational behavior of this system is described by the following partial differential equation:

$$EI\frac{\partial^4\beta(x,t)}{\partial x^4} - N\frac{\partial^2\beta(x,t)}{\partial x^2} + \mu\frac{\partial^2\beta(x,t)}{\partial t^2} - \mu_0 R^0\frac{\partial^4\beta(x,t)}{\partial x^2\partial t^2} + \delta(x)\beta(x,t) = Q(x,t) \tag{1}$$

With

$$Q(x,t) = \frac{1}{\epsilon}\left[-Mg - M\frac{d^2\beta}{dx^2}\right]\left[H\left(x-\xi+\frac{\epsilon}{2}\right) - H\left(x-\xi-\frac{\epsilon}{2}\right)\right] \tag{2}$$

where

x	=	spatial coordinate
t	=	Time
$\beta(x,t)$	=	deflection of the beam
E	=	Young's modulus
I	=	moment of inertia of the beam's cross section about the neutral axis
δ	=	Foundation Constant
μ	=	Area per unit length of the cross section of the beam
μ_0	=	Shear Coefficient
$R0$	=	Radius of Gyration
N	=	Axial Force

The differential operator $\frac{d^2\beta(x,t)}{dx^2}$ is defined as

$$\frac{d^2\beta(x,t)}{dx^2} = \frac{\partial^2\beta(x,t)}{\partial t^2} + 2V\frac{\partial^2\beta(x,t)}{\partial x\partial t} + V^2\frac{\partial^2\beta(x,t)}{\partial x^2} \tag{3}$$

H(.) is the heavy-side function such that

$$H\left(x-\xi+\frac{\epsilon}{2}\right) - H\left(x-\left(\xi-\frac{\epsilon}{2}\right)\right) = \begin{cases} 0 & x < \xi - \frac{\epsilon}{2} \\ 1 & x > \xi - \frac{\epsilon}{2} \end{cases}$$

$$\left(x-\xi-\frac{\epsilon}{2}\right) - H\left(x-\left(\xi+\frac{\epsilon}{2}\right)\right) = \begin{cases} 0 & x < \xi + \frac{\epsilon}{2} \\ 1 & x > \xi + \frac{\epsilon}{2} \end{cases} \tag{4}$$

Hence, the governing equation becomes

$$EI\frac{\partial^4\beta(x,t)}{\partial x^4} - N\frac{\partial^2\beta(x,t)}{\partial x^2} + \mu\frac{\partial^2\beta(x,t)}{\partial t^2} - \mu_0 R^0\frac{\partial^4\beta(x,t)}{\partial x^2\partial t^2} + \delta(x)\beta(x,t)$$
$$= \frac{1}{\epsilon}\left[-Mg - \left(M\frac{\partial^2\beta(x,t)}{\partial x^2} + 2MV\frac{\partial^2\beta(x,t)}{\partial x\partial t} + MV^2\frac{\partial^2\beta(x,t)}{\partial x^2}\right)\right]\left[H\left(x-\xi+\frac{\epsilon}{2}\right) - H\left(x-\xi-\frac{\epsilon}{2}\right)\right] \tag{5}$$

With the boundary conditions

$$\beta(0,t) = 0 = \beta(l,t) \tag{6}$$

$$\frac{\partial^2\beta(0,t)}{\partial x^2} = 0 = \frac{\partial^2\beta(l,t)}{\partial x^2} \tag{7}$$

Without loss of generality, one can consider the initial conditions of the form

$$\beta(x,0) = \frac{\partial\beta(x,0)}{\partial t} \tag{8}$$

Method of Solution

Assume a solution such that the transverse vibration of the beam may be expressed in the following series form

$$\beta(x,t) = \sum_{j=1}^{\infty} X_j(x)T_j(t) \tag{9}$$

$$\frac{\partial\beta(x,t)}{\partial x} = \sum_{j=1}^{\infty} X_j(x)T_j(t)$$
$$\lambda(x,t) = \sum_{j=1}^{\infty} X_j(x)T_j(t) \tag{10}$$
$$\frac{\partial\beta(x,t)}{\partial x} = \sum_{j=1}^{\infty} X_j'(x)T_j(t)$$

$$\frac{\partial^2\beta(x,t)}{\partial x^2} = \sum_{j=1}^{\infty} X_j''(x)T_j(t)$$

$$\frac{\partial^3\beta(x,t)}{\partial x^3} = \sum_{j=1}^{\infty} X_j'''(x)T_j(t)$$
$$\frac{\partial^4\beta(x,t)}{\partial x^4} = \sum_{j=1}^{\infty} X_j^{iv}(x)T_j(t) \tag{11}$$

$$\frac{\partial\beta(x,t)}{\partial t} = \sum_{j=1}^{\infty} X_j(x)\dot{T}_j(t)$$

$$\frac{\partial^2\beta(x,t)}{\partial t^2} = \sum_{j=1}^{\infty} X_j(x)\ddot{T}_j(t) \tag{12}$$

Substituting (9) into (5), we have

$$EI\sum_{i=1}^{\infty} X_i^{iv}(x)T_i(t) + \rho A\sum_{i=1}^{\infty} X_i(x)T''(t) - \mu_0 R^0\sum_{i=1}^{\infty} X_i''(x)T_i''(t) + \sum_{i=1}^{\infty}\delta(x)X_j(x)\,T_j(t)$$
$$-N\sum_{i=1}^{\infty} X_i''(x)T_i(t) = \left[-\frac{Mg}{\epsilon} - \frac{M}{\epsilon}\sum_{i=1}^{\infty} T_i''(t)X_i(x) - \frac{2MV}{\epsilon}\sum_{i=1}^{\infty} T_i'(t)X_i'(x)\right.$$
$$\left.-\frac{V^2M}{\epsilon}\sum_{i=1}^{\infty} T_i(t)X_i''(x)\right]\left\{H\left(x-\xi+\frac{\epsilon}{2}\right) - H\left(x-\xi-\frac{\epsilon}{2}\right)\right\} \tag{13}$$

Furthermore, the forcing term *f*(*x,t*) defined in equation (2) can also be expressed as

$$f(x,t) = \sum_{i=1}^{\infty} T_{fi}(t)X_i(x) \tag{14}$$

Substituting (14) into (2), we have

$$\sum_{i=1}^{\infty} T_{fi}(t)X_i(x) = -\frac{Mg}{\epsilon}\left\{H\left(x-\xi+\frac{\epsilon}{2}\right) - H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$
$$-\frac{M}{\epsilon}\sum_{i=1}^{\infty} T_i''(t)X_i(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right) - H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$
$$-\frac{2MV}{\epsilon}\sum_{i=1}^{\infty} T_i'(t)X_i'(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right) - H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$
$$-\frac{V^2M}{\epsilon}\sum_{i=1}^{\infty} T_i(t)X_i''(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right) - H\left(x-\xi-\frac{\epsilon}{2}\right)\right\} \tag{15}$$

To normalize equation (15), we multiply all through by $Xj(x)$ to obtain

$$\sum_{i=1}^{\infty} T_{fi}(t)X_i(x)\,X_j(x) = -\frac{Mg}{\epsilon}X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right) - H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$
$$-\frac{M}{\epsilon}\sum_{i=1}^{\infty} T_i''(t)X_i(x)X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right) - H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$

$$-\frac{2MV}{\epsilon}\sum_{i=1}^{\infty}T_i'(t)X_i'(x)X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$
$$-\frac{V^2M}{\epsilon}\sum_{i=1}^{\infty}T_i(t)X_i''(x)X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\} \quad (16)$$

Integrating both sides of (16) with respect to x along the length L of the beam, we have

$$\sum_{i=1}^{\infty}T_{fi}(t)X_i(x)\,X_j(x)dx=-\frac{Mg}{\epsilon}X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$-\frac{M}{\epsilon}\sum_{i=1}^{\infty}T_i''(t)\int_0^L X_i(x)X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$-\frac{2MV}{\epsilon}\sum_{i=1}^{\infty}T_i'(t)\int_0^L X_i'(x)X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$-\frac{V^2M}{\epsilon}\sum_{i=1}^{\infty}T_i(t)\int_0^L X_i''(x)X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$(17)$$

From (17), we assume the following

$$0=-\frac{Mg}{\epsilon}\int_0^L\int_0^L X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx \quad (18)$$

Integrating by part using

$$H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\int_0^L X_j(x)dx$$
$$-\int_0^L\int_0^L X_j(x)\left[H'\left(x-\xi+\frac{\epsilon}{2}\right)-H'\left(x-\xi-\frac{\epsilon}{2}\right)\right]dxdx \quad (19)$$

Since $H'(x)=\delta(x)$

$$F(x) = H(x)\; where\; H'(x) - H(x) = \delta(x) \quad (20)$$

such that we get,

$$\text{Let}\quad A_{11}=-\frac{Mg}{\epsilon}\int_0^L X_j\left(\xi+\frac{\epsilon}{2}\right)d\epsilon-\int_0^L X_j\left(\xi-\frac{\epsilon}{2}\right)\epsilon\delta(x) \quad (21)$$

Furthermore, expanding using Taylor series, we obtain,

$$X_j\left(\xi+\frac{\epsilon}{2}\right)=X_j(\xi)+\frac{\left(\frac{\epsilon}{2}\right)}{1!}X_j'(\xi)+\frac{\left(\frac{\epsilon}{2}\right)^2}{2!}X_j''(\xi)+\frac{\left(\frac{\epsilon}{2}\right)^3}{3!}X_j'''(\xi)+\cdots \quad (22)$$

Also,

$$X_j\left(\xi-\frac{\epsilon}{2}\right)=X_j(\xi)-\frac{\left(\frac{\epsilon}{2}\right)}{1!}X_j'(\xi)-\frac{\left(\frac{\epsilon}{2}\right)^2}{2!}X_j''(\xi)-\frac{\left(\frac{\epsilon}{2}\right)^3}{3!}X_j'''(\xi)+\cdots \quad (23)$$

By substituting (22)−(23) into (21), we have

$$A_{11}=-\frac{Mg}{\epsilon}\int_0^L\int_0^L\left[X_j(\xi)+\frac{\left(\frac{\epsilon}{2}\right)}{1!}X_j'(\xi)+\frac{\left(\frac{\epsilon}{2}\right)^2}{2!}X_j''(\xi)+\frac{\left(\frac{\epsilon}{2}\right)^3}{3!}X_j'''(\xi)\right.$$

$$(24)$$

$$-X_j(\xi)-\frac{\left(\frac{\epsilon}{2}\right)}{1!}X_j'(\xi)-\frac{\left(\frac{\epsilon}{2}\right)^2}{2!}X_j''(\xi)-\frac{\left(\frac{\epsilon}{2}\right)^3}{3!}X_j'''(\xi)\Bigg]$$

We get,

$$\epsilon X_j'(\xi)+\frac{\epsilon^3}{24}X_j'''(\xi) \tag{25}$$

Substituting (25) into (19) and having satisfy the condition (4), we have,

$$0=-Mg\left[X_j(\xi)+\frac{\epsilon^2}{24}X_j''(\xi)\right] \tag{26}$$

Similar arguments is applicable to second, third and fourth definite integral in (17), hence, evaluating the integrals using Taylor's series expansion and applying orthogonality properties of the characteristics function $Tfi(t)$ the left hand side of (17), we finally obtain

$$\begin{aligned}Tfi(t)=-Mg\left[X_j(\xi)+\frac{\epsilon^2}{24}X_j''(\xi)\right]-\ M\sum_{j=1}^{\infty}T_i''(t)\Big[X_j(\xi)X_i(\xi)\\ +\frac{\epsilon^2}{24}\Big(X_j(\xi)X_i''(\xi)+2X_j'(\xi)X_i'(\xi)+X_j''(\xi)X_i(\xi)\Big)\Big]\\ -2MV\sum_{j=1}^{\infty}T_i'(t)\left[X_j(\xi)X_i'(\xi)+\frac{\epsilon^2}{24}\Big(X_j(\xi)X_i''(\xi)+2X_j'(\xi)X_i''(\xi)+X_j''(\xi)X_i(\xi)\Big)\right]\\ -V^2M\sum_{j=1}^{\infty}T_i'(t)\left[X_j(\xi)X_i''(\xi)+\frac{\epsilon^2}{24}\Big(X_j(\xi)X_i^{iv}(\xi)+2X_j'(\xi)X_i'''(\xi)+X_j''(\xi)X_i''(\xi)\Big)\right]\end{aligned}$$

Furthermore, from (13) we have that

$$EI\sum_{i=1}^{\infty}X_i^{iv}(x)T_i(t)+\rho A\sum_{i=1}^{\infty}X_i(x)T''(t)-\mu_0R^0\sum_{i=1}^{\infty}X_i''(x)T_i''(t)+\sum_{j=1}^{\infty}\delta(x)X_j(x)\,T_j(t) \tag{27}$$

$$-N\sum_{i=1}^{\infty}X_i''(x)T_i(t)=\sum_{i=1}^{\infty}T_{fi}(t)X_i(x) \tag{28}$$

substituting (27) into (28) becomes

$$\begin{aligned}EI\sum_{i=1}^{\infty}X_i^{iv}(x)T_i(t)+\rho A\sum_{i=1}^{\infty}X_i(x)T''(t)-\mu_0R^0\sum_{i=1}^{\infty}X_i''(x)T_i''(t)+\sum_{j=1}^{\infty}\delta(x)X_j(x)\,T_j(t)\\ -N\sum_{i=1}^{\infty}X_i''(x)T_i(t)=-Mg\left[X_j(\xi)+\frac{\epsilon^2}{24}X_j''(\xi)\right]-\ M\sum_{j=1}^{\infty}T_i''(t)\Big[X_j(\xi)X_i(\xi)\\ +\frac{\epsilon^2}{24}\Big(X_j(\xi)X_i''(\xi)+2X_j'(\xi)X_i'(\xi)+X_j''(\xi)X_i(\xi)\Big)\Big]\\ -2MV\sum_{j=1}^{\infty}T_i'(t)\left[X_j(\xi)X_i'(\xi)+\frac{\epsilon^2}{24}\Big(X_j(\xi)X_i'''(\xi)+2X_j'(\xi)X_i''(\xi)+X_j''(\xi)X_i(\xi)\Big)\right]\\ -V^2M\sum_{j=1}^{\infty}T_i'(t)\left[X_j(\xi)X_i''(\xi)+\frac{\epsilon^2}{24}\Big(X_j(\xi)X_i^{iv}(\xi)+2X_j'(\xi)X_i'''(\xi)+X_j''(\xi)X_i''(\xi)\Big)\right]\end{aligned} \tag{29}$$

so that (29) becomes

$$EI\sum_{i=1}^{\infty}X_i^{iv}(x)T_i(t)+\rho A\sum_{i=1}^{\infty}X_i(x)T''(t)-\mu_0R^0\sum_{i=1}^{\infty}X_i''(x)T_i''(t)+\sum_{j=1}^{\infty}\delta(x)X_j(x)\,T_j(t)$$

$$-N\sum_{i=1}^{\infty}X_i''(x)T_i(t)-Mg\left[X_j(\xi)+\frac{\epsilon^2}{24}X_j''(\xi)\right]+M\sum_{j=1}^{\infty}T_i''(t)\Big[X_j(\xi)X_i(\xi)+\frac{\epsilon^2}{24}\Big(X_j(\xi)X_i''(\xi)+2X_j'(\xi)X_i'(\xi)+X_j''(\xi)X_i(\xi)\Big)\Big]$$
$$+2MV\sum_{j=1}^{\infty}T_i'(t)\left[X_j(\xi)X_i'(\xi)+\frac{\epsilon^2}{24}\Big(X_j(\xi)X_i'''(\xi)+2X_j'(\xi)X_i''(\xi)+X_j''(\xi)X_i(\xi)\Big)\right]$$
$$+V^2M\sum_{j=1}^{\infty}T_i'(t)\left[X_j(\xi)X_i''(\xi)+\frac{\epsilon^2}{24}\Big(X_j(\xi)X_i^{iv}(\xi)+2X_j'(\xi)X_i'''(\xi)+X_j''(\xi)X_i''(\xi)\Big)\right]=0$$

For the boundary conditions given under the governing equation

$$X_i(x)=sin\left[\frac{i\pi x}{L}\right] \tag{31}$$

To obtain a set of exact governing differential equation for the simply supported beam under consideration, we substitute (31) into (17) to obtain

$$\sum_{i=1}^{\infty}T_{fi}(t)\int_0^L sin\left[\frac{i\pi}{L}x\right]sin\left[\frac{j\pi}{L}x\right]dx$$
$$=-M\frac{g}{\epsilon}\int_0^L sin\frac{i\pi}{L}x\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$-2M\frac{V}{\epsilon L}\sum_{i=1}^{\infty}T_i'(t)\int_0^L sin\frac{i\pi}{L}x sin\frac{i\pi}{L}x\ \left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$-4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}T_i'(t)\int_0^L sin\frac{i\pi}{L}x cos\frac{i\pi}{L}x\ \left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}T_i(t)\int_0^L sin\frac{i\pi}{L}x sin\frac{i\pi}{L}x\ \left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx \tag{32}$$

Evaluating the above integrals, we have

$$\alpha_{11}=sin\frac{j\pi}{L}x\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx=2sin\left[\frac{j\pi}{L}\xi\right]sin\left[\frac{i\pi}{L}\epsilon\right] \tag{33}$$

$$\alpha_{12}=\int_0^L sin\frac{\pi}{L}x sin\frac{\pi}{L}x\ \left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$=\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j) \tag{34}$$

$$\alpha_{13}=\int_0^L sin\frac{\pi}{L}x cos\frac{\pi}{L}x\ \left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$=\frac{1}{i-j}\ sin\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\ sin\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j) \tag{35}$$

$$\alpha_{14}=\int_0^L sin\frac{\pi}{L}x sin\frac{\pi}{L}x\ \left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$=\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j) \tag{36}$$

$$Q_5=\int_0^L sin\left[\frac{i\pi}{L}x\right]sin\left[\frac{j\pi}{2L}x\right]dx=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases} \tag{37}$$

By substituting equations (33)−(37) into (32), we have

$$T_{fi}(t)=\ sin\left[\frac{i\pi}{L}\xi\right]sin\left[\frac{j\pi}{L}\epsilon\right]$$

$$-2\frac{M}{\epsilon L}\sum_{i=1}^{\infty}T_i''(t)\left\{\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right\}$$
$$-4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}T_i'(t)\left\{\frac{1}{i+j}\ sin\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)-\frac{1}{i-j}\ sin\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)\right\}$$
$$-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}T_i(t)\left\{\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right\}$$

$i \neq j, i = 1, 2, 3, \ldots$

By replacing the right hand side of (**??**) with the right hand side of (38), we finally obtain

$$EI\left(\frac{\pi}{L}\right)^4 i^4T_i(t)+\rho AT_i''(t)-\mu_0R^0\left(\frac{\pi}{L}\right)^2 i^2T_i''(t)-N\left(\frac{\pi}{L}\right)^2 T_i(t)+\delta T_j(t)$$
$$=-M\frac{g}{i\pi\epsilon}\ sin\left[\frac{i\pi}{L}\xi\right]sin\left[\frac{j\pi}{2L}\epsilon\right]$$
$$-2\frac{M}{\epsilon L}\sum_{i=1}^{\infty}T_i''(t)\left\{\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right\}$$
$$-4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}T_i'(t)\left\{\frac{1}{i+j}\ sin\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)-\frac{1}{i-j}\ sin\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)\right\}$$
$$-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}T_i(t)\left\{\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right\} \quad (39)$$

$i \neq j, i = 1, 2, 3, \ldots$

$$EI\frac{\pi}{L^4}^4 i^4T_i(t)+\rho AT_i''(t)-\rho A\sigma_G^2\frac{\pi}{L^2}^2\left(1+\frac{E}{KG}\right)i^2T_i''(t)+\frac{\rho^2A\sigma_G^2}{KG}T_i^{iv}$$
$$=-M\frac{g}{i\pi\epsilon}\ sin\left[\frac{i\pi}{L}\xi\right]sin\left[\frac{j\pi}{2L}\epsilon\right]-\frac{2M}{\epsilon L}\sum_{i=1}^{\infty}T_i''(t)\left\{\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)\right\}$$
$$+\frac{2M}{\epsilon L}\sum_{i=1}^{\infty}T_i''(t)\left\{\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right\}$$
$$-4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}T_i'(t)\left\{\frac{1}{i+j}\ sin\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right\}$$
$$+4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}T_i'(t)\left\{\frac{1}{i-j}\ sin\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)\right\}$$
$$-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}T_i(t)\left\{\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)\right\}$$
$$-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}T_i(t)\left\{\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right\}$$

$sin\frac{\pi}{2L}\epsilon(i+j)\Big\}\ i = 1, 2, 3, \ldots, i \neq j$

$$EI\frac{\pi}{L^4}^4 i^4T_i(t)+\rho AT_i''(t)-\rho A\sigma_G^2\frac{\pi}{L^2}^2\left(1+\frac{E}{KG}\right)i^2T_i''(t)+\frac{\rho^2A\sigma_G^2}{KG}T_i^{iv} \quad (40)$$
$$=-M\frac{g}{i\pi\epsilon}\ sin\left[\frac{i\pi}{L}\xi\right]sin\left[\frac{j\pi}{2L}\epsilon\right]-\frac{2M}{\epsilon L}\sum_{i=1}^{\infty}T_i''(t)\left\{\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)\right\}$$
$$+\frac{2M}{\epsilon L}\sum_{i=1}^{\infty}T_i''(t)\left\{\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right\}$$

$$
\begin{aligned}
&-4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}T_i'(t)\left\{\frac{1}{i+j}\ sin\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right\}\\
&+4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}T_i'(t)\left\{\frac{1}{i-j}\ sin\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)\right\}\\
&-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}T_i(t)\left\{\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)\right\}\\
&-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}T_i(t)\left\{\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right\}
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
&\left\{\rho A-\mu_0R^0\left(\frac{\pi}{L}\right)^2+\frac{2M}{\epsilon L}\left[\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right]\right\}T_i''(t)\\
&+\left\{2MV\frac{i\pi}{\epsilon L^2}h\left[\frac{1}{i+j}\ sin\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)-\frac{1}{i-j}\ sin\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)\right]\right\}T_i'(t)\\
&\left\{EI\frac{\pi^4}{L^4}\ i^4-N\left(\frac{\pi}{L}\right)^2j^2+\delta-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\left[\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)\right.\right.\\
&\left.\left.-\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right]\right\}T_i(t)=-2M\frac{g}{i\pi\epsilon}\ sin\left[\frac{i\pi}{L}\xi\right]sin\left[\frac{j\pi}{2L}\epsilon\right]
\end{aligned}
\tag{42}
$$

Numerical Analysis

The numerical method alluded to is the central difference technique applying the central difference formula to the derivative in equation (**??**), we obtain

$$T_j''(t)=\frac{T_{j+1}-2T_j+T_{j-1}}{h^2} \tag{43}$$

and

$$T_j'(t)=\frac{T_{j+1}-T_{j-1}}{2h} \tag{44}$$

Substituting equations (43) and (44) into equation (42) ,

$$
\begin{aligned}
&\left\{\rho A-\mu_0R^0\left(\frac{j\pi}{L}\right)^2+\frac{2M}{\epsilon L}\left[\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)\right.\right.\\
&\left.\left.-\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right]\right\}\frac{T_{j+1}-2T_j+T_{j-1}}{h^2}\\
&+\left\{2MV\frac{i\pi}{\epsilon L^2}h\left[\frac{1}{i+j}\ sin\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)-\frac{1}{i-j}\ sin\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)\right]\right\}\frac{T_{j+1}-T_{j-1}}{2h}\\
&\left\{EI\frac{\pi^4}{L^4}\ i^4-N\left(\frac{\pi}{L}\right)^2j^2+\delta-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\left[\frac{1}{i-j}\ cos\frac{\pi}{L}\xi(i-j)sin\frac{\pi}{2L}\epsilon(i-j)\right.\right.\\
&\left.\left.-\frac{1}{i+j}\ cos\frac{\pi}{L}\xi(i+j)sin\frac{\pi}{2L}\epsilon(i+j)\right]\right\}T_i(t)=-2M\frac{g}{i\pi\epsilon}\ sin\left[\frac{i\pi}{L}\xi\right]sin\left[\frac{j\pi}{2L}\epsilon\right]
\end{aligned}
\tag{45}
$$

further simplification gives

$$T_{j+1}=\frac{4G}{2G+Ph}T_j-\frac{2G-Ph}{2G+Ph}T_{j-1}-\frac{h^2Q}{2G+Ph}T_j+\frac{h^2}{2G+Ph}F \tag{46}$$

Where

$$G = \rho A - \mu_0 R^0 \left(\frac{j\pi}{L}\right)^2 + \frac{2M}{\epsilon L}\left[\frac{1}{i-j} \cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j) - \frac{1}{i+j} \cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right]$$

$$P = 2MV\frac{i\pi}{\epsilon L^2}\left[\frac{1}{i+j} \sin\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j) - \frac{1}{i-j} \sin\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)\right]$$

$$Q = EI\frac{\pi^4}{L^4} i^4 - N\left(\frac{\pi}{L}\right)^2 j^2 + \delta - 2MV^2\frac{i^2\pi^2}{\epsilon L^3}\left[\frac{1}{i-j} \cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j) - \frac{1}{i+j} \cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right]$$

$$F = -2M\frac{gL}{i\pi\epsilon}\sin\left[\frac{i\pi}{L}\xi\right]\sin\left[\frac{j\pi}{2L}\epsilon\right] \tag{47}$$

RESULTS AND DISCUSSION

The following example illustrates the analysis carried out in this paper. Beam dimension and specification:

The beam was made of steel E=2.10×10^{11}N

Length(L)=10 m

Density of the mass (ρ) = 1.64 × 10^{8}kg/m^3

Surface area of the beam cross section A=6×10^{-6}m^2

Shear coefficient K=0.5, 1.0, 1.5

Shear modulus G=0.2, 0.4, 0.8

Distance covered by the load ξ=0.1, 0.2, 0.3

Load's lengthϵ=0.12, 1.24, 3.60

Modulus of shear foundation σ_G=0.1, 0.2, 0.3

Rigidity of the Beam EI = 1.74 × 10^{-5}m^4

DISCUSSION

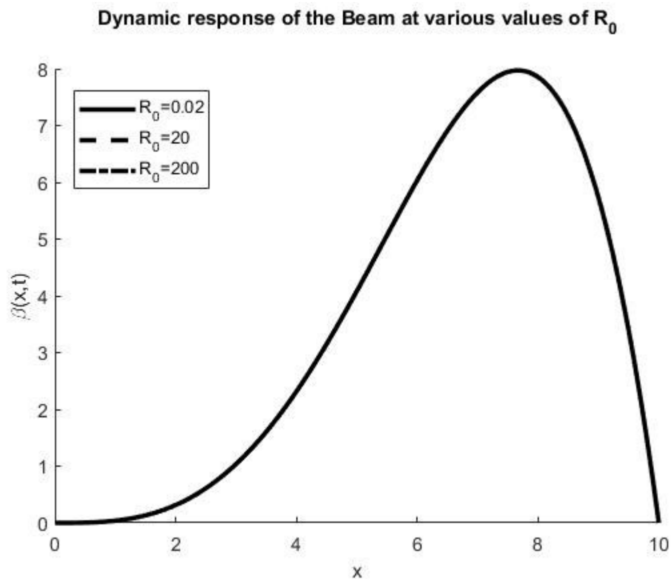
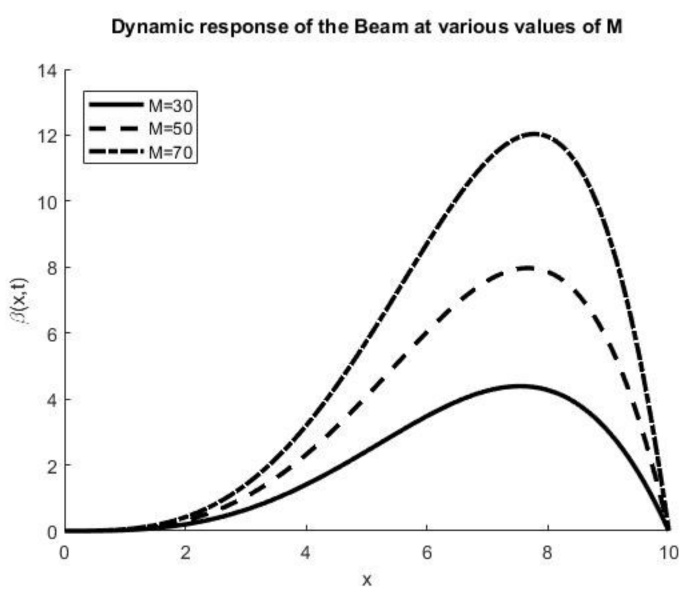


Figure 1: The effect of R_0 on the deflection of beam

Figure 1 displays the dynamic response of the beam at various values of R_0. It is observed that it first move in a steady state before deflecting and the deflected amplitude increases as R_0 increases.



2: *M*

Figure 2 shows the dynamic response of the beam at various values of mass of the load. It is observed that the vibration of the beam initially moves in a steady state before deflecting and the deflected amplitude increases as the mass of the load increases.

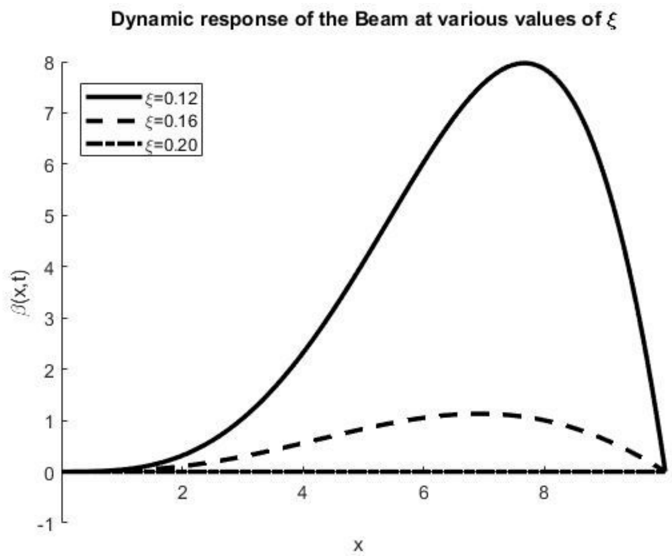
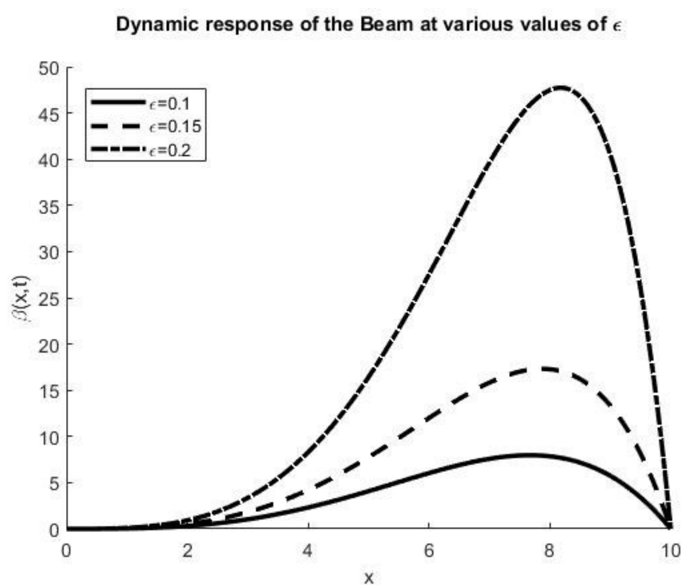


Figure 3: The effect of ξ on the deflection of beam

Figure 3 displays the dynamic response of the beam at various values of the length of the load. It is observed that the vibration of the beam initially moves in a steady state before deflecting and the deflected amplitude decreases as the length of the load increases.



4:

Figure 4 shows the dynamic response of the beam at various values of the distance covered by the load. It is observed that the vibration of the beam initially moves in a steady state before deflecting and the deflected amplitude increases as the distance covered by the load increases.

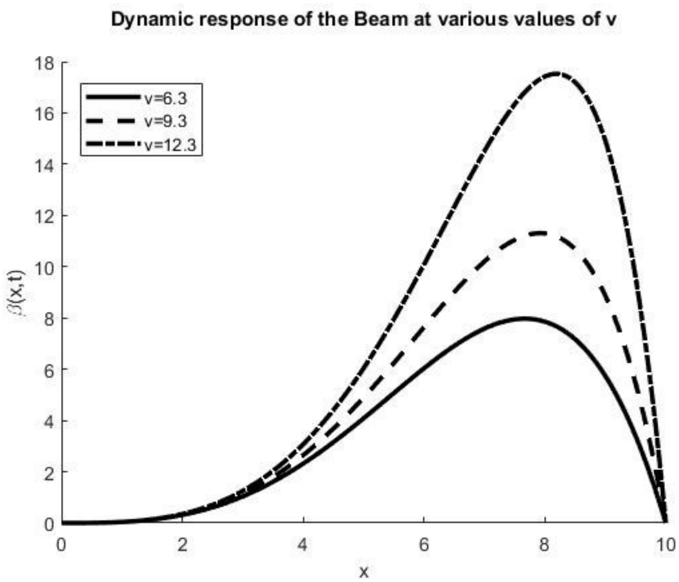


Figure 5: The effect of v on the deflection of beam

Figure 5 shows the dynamic response of the beam at various values of the speed of the load. It is observed that the vibration of the beam initially moves in a steady state before deflecting and the deflected amplitude increases as the speed at which the load moves increases.

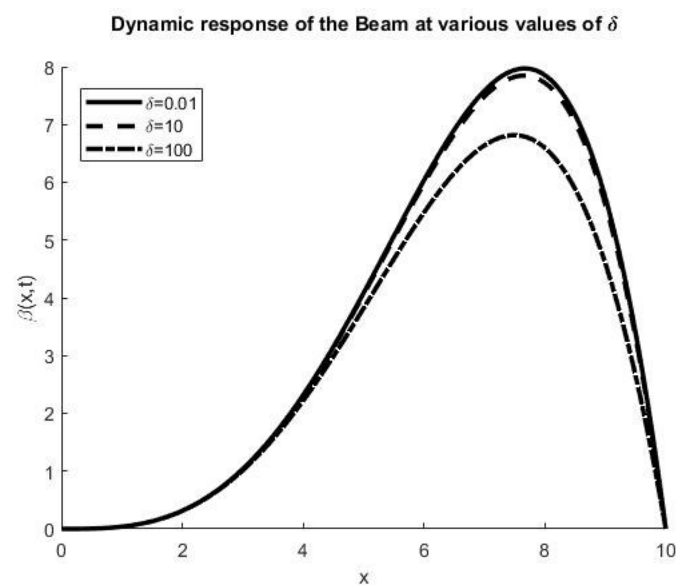


Figure 6: The effect of δ on the deflection of beam

Figure 6 shows the dynamic response of the beam at various values of δ. It is observed that the vibration of the beam initially moves in a steady state before deflecting and the deflected amplitude decreases as δ increases.

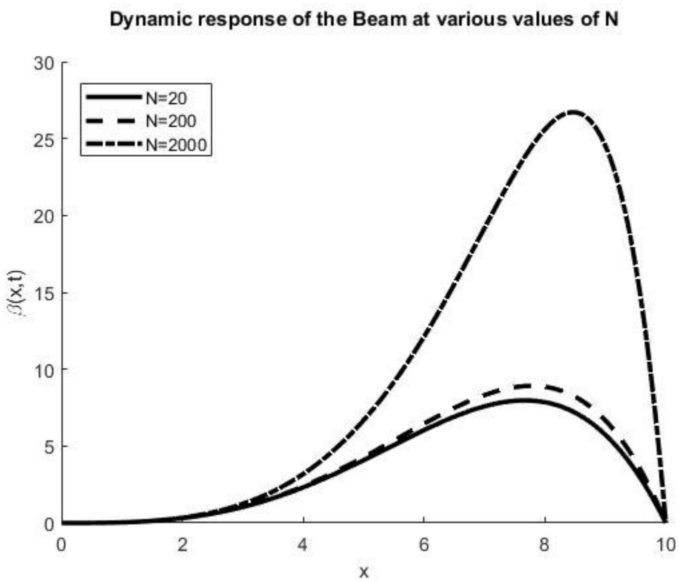


Figure 7: The effect of N on the deflection of beam

Figure 7 shows the dynamic response of the beam at various values of the axial force. It is observed that the deflection of the beam initially moves in a steady state before deflecting and the deflected amplitude increases as the axial force on the beam increases.

CONCLUSION

Dynamic response of a Rayleigh beam was considered in this project work. The governing equation of fourth order partial differential equation was reduced to a fourth order ordinary differential equation by normalising the governing equation. The reduced second order equation was solved using finite difference method. The deflection for various parameters of the beam was considered and was plotted against x using a computer program (MATLAB).

It can be concluded from the numerical results that the dynamic response of the beam initially moves in a steady state before deflecting and the deflected amplitude increases as the axial force, distance covered by the load, Mass of the load, speed at which the load moves increases but decreases as the length of the load, coefficient of the foundation increases. This study also conclude that R_0 has no/little effects on the structure of the beam.

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